DIFFERENTIAL EQUATIONS FOR THE STUDY OF RADIATIVE HEAT EXCHANGE IN AN ABSORBING-EMITTING MEDIUM

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Exact differential equations have been derived for the sums and differences of hemispherical radiative flows in an absorbing-emitting medium. A method is given for the solution of these equations for a flat layer.

In connection with the complexity of solving the integral equations derived in determining the temperatures in absorbing-emitting media with energy sources, the differential methods of investigation are of great interest. The Schuster-Schwarzschild method—known also as the "forward-backward" approximation—was historically the first. The essence of this method involves the following. The equations for the transfer of radiation for opposing hemispheres in the form [1]

$$\cos\vartheta\, I'_+(\tau,\ \vartheta) = -I_+(\tau,\ \vartheta) + B(\tau),$$
$$\cos\vartheta\, I'_-(\tau,\ \vartheta) = I_-(\tau,\ \vartheta) - B(\tau) \qquad (1)$$

are integrated over the solid angles within the limits of the correspoding hemispheres (here, and in the following, the derivatives (') are defined according to the optical distance τ. The quantity B is understood to mean $\sigma T^4/\pi$ for a noncontinuous radiation spectrum (in a "gray" medium) or the Planck distribution function for monochromatic radiation). As a result, we derive equations for hemispherical radiation flows:

$$q'_+(\tau) = -m_+(\tau)\, q_+(\tau) + 2\pi B(\tau),$$
$$q'_-(\tau) = m_-(\tau)\, q_-(\tau) - 2\pi B(\tau), \qquad (2)$$

where the auxiliary functions $m_+(\tau)$ and $m_-(\tau)$ are defined as follows [2]:

$$m_+(\tau) = \frac{\int\limits_{(+2\pi)} I_+ d\omega}{\int\limits_{(+2\pi)} I_+ \cos\vartheta\, d\omega},$$
$$m_-(\tau) = \frac{\int\limits_{(-2\pi)} I_- d\omega}{\int\limits_{(-2\pi)} I_- \cos\vartheta\, d\omega}. \qquad (3)$$

Although the functions m_+ and m_- depend on τ and are not equal to each other, in the subject approximation it is assumed that $m_+ = m_- = m = \text{const}$, which makes it possible in system (2) to isolate the resulting radiant flux $q = q_+ - q_-$, characterizing the distribution of the heat sources or sinks. In this case, with the help of (2), the problem of the temperature of the medium (B) for a given distribution q (or a given relationship between B and q) is easily solved. However, this solution may be regarded as approximate.

As shown in [1, 2], Eqs. (2) with the values $m_+ = m_- = m = 2$ (which corresponds to the assumption in (3) that I_+ and I_- are independent of ϑ) yield results close to reality only for a small optical thickness of the layer. With a great optical thickness they yield results accurate to 25%. A more exact solution of (2) (with refinement of the values of $m_+(\tau)$ and $m_-(\tau)$ in (3) according to the approximate solution) yields no significant improvement in the results, as is noted at the end of the article, in the solution of a specific problem.

A better system for the study of radiative heat exchange in optically dense media is the system of equations for the resulting radiant flux q and radiation density V [2, 3]

$$cV'(\tau) = -\alpha(\tau)\, q(\tau), \quad q'(\tau) = -cV(\tau) + 4\pi B(\tau), \quad (4)$$

where the auxiliary function is given by

$$\alpha(\tau) = \frac{\int\limits_{(+2\pi)} I'_+ d\omega + \int\limits_{(-2\pi)} I'_- d\omega}{\int\limits_{(+2\pi)} I'_+ \cos^2\vartheta\, d\omega + \int\limits_{(-2\pi)} I'_- \cos^2\vartheta\, d\omega}. \qquad (5)$$

Even in the first, so-called diffusion approximation (with a constant value for $\alpha = 3$), Eqs. (4) make it possible to derive virtually exact results deep within optically dense media. However, near the boundaries or in optically thin layers the results obtained in this manner are approximate. To find a more exact solution of the problem near the boundary (with refinement of $\alpha(\tau)$ according to the first approximation) the utilization of Eqs. (4) leads to difficulties. The problem here lies in the fact that the value of $\alpha(\tau)$ at the boundary tends toward infinity if, as will be demonstrated later on, a discontinuity in the temperature values occurs at this boundary.

Here we have derived equations which include the magnitude of the resulting radiant flux and the auxiliary functions which do not become infinite at the boundaries. In conjunction with these it becomes possible to derive excellent results both within the depths of optically dense media and near the boundaries, because of the possibility of solving the equations by the method of successive approximations.

Adding and subtracting Eqs. (2), while denoting $p = q_+ + q_-$ and $q = q_+ - q_-$, we obtain

$$p'(\tau) = -\beta(\tau)\, q(\tau), \quad q'(\tau) = -\gamma(\tau)\, p(\tau) + 4\pi B(\tau), \quad (6)$$

where

$$\beta(\tau) = \frac{\int\limits_{(+2\pi)} I_+ d\omega - \int\limits_{(-2\pi)} I_- d\omega}{\int\limits_{(+2\pi)} I_+ \cos\vartheta\, d\omega - \int\limits_{(-2\pi)} I_- \cos\vartheta\, d\omega};$$

$$\gamma(\tau) = \frac{\int\limits_{(+2\pi)} I_+ d\omega + \int\limits_{(-2\pi)} I_- d\omega}{\int\limits_{(+2\pi)} I_+ \cos\vartheta\, d\omega + \int\limits_{(-2\pi)} I_- \cos\vartheta\, d\omega}. \quad (7)$$

Let us formulate the boundary conditions for (6) by means of the following relationships which are valid for diffusely reflecting surfaces [2]:

$$q(b) = q_+(b) - q_-(b) = \pm A_b [q_\pm(b) - \pi B_b], \quad (8)$$

where A_b is the emissivity of the bounding surface, B_b is the radiation intensity of the surface for $A_b = 1$, the upper signs (+) correspond to the case in which the vector τ (or q_+) is directed toward the boundary surface, and the lower signs (−) correspond to the opposite direction. Using the adopted denotations for p and q, we transform (8) to the corresponding form

$$p(b) = 2\pi B_b \pm \frac{2 - A_b}{A_b} q(b). \quad (9)$$

The exact solution for the problem of the temperature of the medium with the given distribution $q(\tau)$, according to (6) and boundary conditions (9), has the form

$$4\pi B(\tau) = q'(\tau) + \gamma(\tau)[2\pi B_b \pm \frac{2 - A_b}{A_b} q(b) - \int\limits_b^\tau \beta(\tau) q(\tau) d\tau]. \quad (10)$$

To find the solution of (10) we must know the two functions $\beta(\tau)$ and $\gamma(\tau)$. Their values can be calculated from (7) if the functions $I_+(\tau,\vartheta)$ and $I_-(\tau,\vartheta)$ are known. The latter are solutions of (1):

$$I_+(\tau, \vartheta) = C(\vartheta)\exp(-\tau\sec\vartheta) + \sum_{k=0}^{\infty} (-1)^k \cos^k\vartheta\, B^{(k)}(\tau),$$

$$I_-(\tau, \vartheta) = D(\vartheta)\exp(+\tau\sec\vartheta) + \sum_{k=0}^{\infty} \cos^k\vartheta\, B^{(k)}(\tau),$$

where $C(\vartheta)$ and $D(\vartheta)$ are functions defined from the boundary conditions, while the infinite series are partial solutions of nonuniform equations (1). Subsequently, for simplicity in analyzing the functions β and γ we find $C(\vartheta)$ and $D(\vartheta)$ only when the medium is bounded by two infinite planes with "black" surfaces ($A_b = 1$). The optical distance between the planes is equal to Δ. Here $I_+(0,\vartheta)$ and $I_-(\Delta,\vartheta)$ are, respectively, equal to the wall-radiation intensities B_0 and B_Δ, while the expressions for C and D assume the form

$$C(\vartheta) = B_0 - \sum_{k=0}^{\infty} (-1)^k \cos^k\vartheta\, B^{(k)}(0),$$

$$D(\vartheta) = [B_\Delta - \sum_{k=0}^{\infty} \cos^k\vartheta\, B^{(k)}(\Delta)]\exp(-\Delta\sec\vartheta). \quad (11)$$

Substituting the values of $I_+(\tau,\vartheta)$ and $I_-(\tau,\vartheta)$ with $C(\vartheta)$ and $D(\vartheta)$ according to (11) into (7) and representing $d\omega$ as $2\pi\sin\vartheta\, d\vartheta$ in the latter (for the plane case), we obtain

$$\beta(\tau) = \Big[B_0E_2(\tau) - B_\Delta E_2(\Delta-\tau) - \sum_{k=0}^{\infty}(-1)^k E_{k+2}(\tau)B^{(k)}(0) + \sum_{k=0}^{\infty}E_{k+2}(\Delta-\tau)B^{(k)}(\Delta) - \sum_{k=0}^{\infty}\frac{2}{2k+2}B^{(2k+1)}(\tau)\Big] \times \Big[B_0E_3(\tau) - B_\Delta E_3(\Delta-\tau) - \sum_{k=0}^{\infty}(-1)^k E_{k+3}(\tau)B^{(k)}(0) + \sum_{k=0}^{\infty}E_{k+3}(\Delta-\tau)B^{(k)}(\Delta) - \sum_{k=0}^{\infty}\frac{2}{2k+3}B^{(2k+1)}(\tau)\Big]^{-1},$$

$$\gamma(\tau) = \Big[B_0E_2(\tau) + B_\Delta E_2(\Delta-\tau) - \sum_{k=0}^{\infty}(-1)^k E_{k+2}(\tau)B^{(k)}(0) - \sum_{k=0}^{\infty}E_{k+2}(\Delta-\tau)B^{(k)}(\Delta) + \sum_{k=0}^{\infty}\frac{2}{2k+1}B^{(2k)}(\tau)\Big] \times \Big[B_0E_3(\tau) + B_\Delta E_3(\Delta-\tau) - \sum_{k=0}^{\infty}(-1)^k E_{k+3}(\tau)B^{(k)}(0) - \sum_{k=0}^{\infty}E_{k+3}(\Delta-\tau)B^{(k)}(\Delta) + \sum_{k=0}^{\infty}\frac{2}{2k+2}B^{(2k)}(\tau)\Big]^{-1}, \quad (12)$$

where $E_\nu(x) = \int\limits_1^\infty u^{-\nu}\exp(-xu)\,du$ is an integroexponential function of ν-th order.

Let us analyze the derived expressions for $\beta(\tau)$ and $\gamma(\tau)$, representing $B(\tau)$ in the form of the series

$$B(\tau) = \sum_{k=0}^{n} a_k \tau^k \quad (n \neq 0).$$

It is not difficult to prove that within the layer as $\Delta \to \infty$ the values of β and γ tend independently from the coefficients a_k to constant values of $\beta = 3/2$ and $\gamma = 2$. Indeed, when $\tau \gg 1$ and $\Delta - \tau \gg 1$ in the numerator and the denominator of expressions (12), the term which does not contain the integroexponential function ($E_\nu(x \gg 1) \to 0$) and which, moreover, exhibits the greatest power of τ, is decisive with respect to value. We will refer to these constant values as limiting values.

If we substitute the found values of $I_+(\tau,\vartheta)$ and $I_-(\tau,\vartheta)$ into formula (5), we obtain an expression for $\alpha(\tau)$ analogous to expressions (12)

$$\alpha(\tau) = \Big[B_0 E_1(\tau) - B_\Delta E_1(\Delta-\tau) - \sum_{k=0}^{\infty}(-1)^k E_{k+1}(\tau) B^{(k)}(0) + \sum_{k=0}^{\infty} E_{k+1}(\Delta-\tau) B^{(k)}(\Delta) - \sum_{k=0}^{\infty}\frac{2}{2k+1}B^{(2k+1)}(\tau)\Big] \times \Big[B_0 E_3(\tau) - B_\Delta E_3(\Delta-\tau) - \sum_{k=0}^{\infty}(-1)^k E_{k+3}(\tau) B^{(k)}(0) + \sum_{k=0}^{\infty} E_{k+3}(\Delta-\tau) B^{(k)}(\Delta) - \sum_{k=0}^{\infty}\frac{2}{2k+3}B^{(2k+1)}(\tau)\Big]^{-1}. \quad (13)$$

An analysis of expression (13) makes it possible for us to establish that the limit value of $\alpha = 3$, while the value of α at the boundaries of the layer when $B(b) \neq B_b$ become infinite ($E_\nu(0) = 1/\nu - 1$). This last feature of the function $\alpha(\tau)$ makes it more difficult to use Eqs. (4) to find a solution for them that is more exact than the diffusion approximation.

Analogously, substituting the found values of $I_+(\tau, \vartheta)$ and $I_-(\tau, \vartheta)$ into (3), we can establish that the limit values of m_+ and m_- are equal to 2. At the boundaries of the layer when $B(b) \neq B_b$ the values of m_+ and m_- as well as β and γ are finite.

The special form of the found expressions for $I_+(\tau, \vartheta)$ and $I_-(\tau, \vartheta)$ with the simple boundary conditions does not disrupt the generality of the limit values: the boundary effects do not affect the process of radiant-energy transfer within optically large volumes. Practically, with $\Delta > 10$ the values of α, β, γ, m_+, and m_- within the layer are already close to the limit. Consequently, solution of (6) in first approximation, as well as the solution of (4) in diffusion approximation ($\alpha = 3$), can be derived with limit values of $\beta = 3/2$ and $\gamma = 2$. Equations (4) and (6) here yield identical results. In second approximation $\beta(\tau)$ and $\gamma(\tau)$ are calculated from (12) where $B(\tau)$ corresponds to the solution in first approximation. If the form of $B(\tau)$ in first approximation is sufficiently simple, the calculation of $\beta(\tau)$ and $\gamma(\tau)$, and then of $B(\tau)$ in second approximation, presents no particular difficulty. Here, when all arbitrary $B^{(k)}(\tau)$ are finite in value (i.e., the series in expressions (12) are not broken), the function $B(\tau)$ must be approximated by a simpler function so that the number of terms in (12) is small. If it is impossible to accomplish such an approximation with good accuracy, expressions (12) should be presented in an integral form convenient for numerical integration:

$$\beta(\tau),\ \gamma(\tau) = \Big\{ [B_0 - B(\tau)] E_2(\tau) \mp [B_\Delta - B(\tau)] E_2(\Delta-\tau) + \int_0^\tau [B(x) - B(\tau)] E_1(\tau - x)\,dx \mp \int_\tau^\Delta [B(x) - B(\tau)] E_1(x-\tau)\,dx \Big\} \times \Big\{ [B_0 - B(\tau)] E_3(\tau) \mp [B_\Delta - B(\tau)] E_3(\Delta-\tau) + \int_0^\tau [B(x) - B(\tau)] E_2(\tau - x)\,dx \mp \int_\tau^\Delta [B(x) - B(\tau)] E_2(x-\tau)\,dx \Big\}^{-1}.$$

The sign (−) corresponds to the expression for β and the sign (+) corresponds to the expression for γ.

As an example of a solution for the problem of the temperature of a medium for known $q(\tau)$ let us examine (10) for the case $q = \text{const}$, $\Delta \to \infty$, $B_0 = 0$, $A_0 = 1$ (the problem of the stellar photosphere). With an exact Hopf solution [4] for this problem it becomes possible to check the convergence of the proposed method of solution. For the subject case ($q < 0$) formula (10) assumes the form

$$B(\tau) = -\frac{q}{4\pi}\gamma(\tau)\Big[1 + \int_0^\tau \beta(\tau)\,d\tau\Big]. \quad (14)$$

In first approximation (with limit values of $\beta = 3/2$ and $\gamma = 2$) from the solution of (14) we obtain the result known in astrophysics as the Eddington approximation [1]:

$$B_1(\tau) = -\frac{q}{\pi}\Big(\frac{1}{2} + \frac{3}{4}\tau\Big). \quad (15)$$

The greatest deviation (15.5%) in the quantity $B_1(\tau)$ from its exact value is found when $\tau = 0$: $B_1(0) = -q/2\pi$, the exact value of $B(0) = -(3q)^{1/2}/4\pi$.

Substituting (15) into (12) in which, in this case, terms with derivatives of $B(\tau)$ higher than the first disappear, we obtain expressions for $\beta_2(\tau)$ and $\gamma_2(\tau)$. Substitution of the latter into (14) yields $B_2(\tau)$. The results of the calculations have demonstrated that the values of $\beta_2(\tau)$ and $\gamma_2(\tau)$ vary monotonically from 7/4 (when $\tau = 0$) to their limit values which they virtually attain (with accuracy to 1%) when $\tau = 1$ (the values of integroexponential functions were taken from the tables in [5]).

The maximum divergence between $B_2(\tau)$ and the exact value of $B(\tau)$ is found at the boundary of the layer and amounts to 1%. Moreover, according to the second approximation, $B_2'(0) \to \infty$ (because $\gamma_2'(0) \to \infty$), which coincides with the exact Hopf solution. Thus, the method of successive approximations with use of the initial values of $\beta = 3/2$ and $\gamma = 2$ ensures rapid convergence for the solution of (14).

The solution of the subject problem by means of Eqs. (4) in first (diffusion) approximation, as well as in accordance with Eqs. (6), has the form of (15) [3]. Thus, according to (13), the value of $\alpha_2(0)$ becomes infinite as a result of $B(0) \neq B_0$. The numerical calculations, according to (4), in second approximation therefore involves difficulties associated with their behavior when $\tau = 0$.

The results of the solution for this very problem, according to (2), are not satisfactory, since in any approximation the values of m_+ and m_- with $\tau > 1$ are virtually equal to 2. As is well known [1, 2], the Schuster-Schwarzschild approximation obtained in this case yields results that are accurate to 25%.

In conclusion we present the exact expression for the resulting radiant flux between two parallel diffusely reflecting planes. This can be found by solving only the first of the equations in (6) in accordance with boundary conditions (9). As a result we obtain

$$q = \frac{\pi B_0 - \pi B_\Delta}{\frac{1}{A_0} + \frac{1}{A_\Delta} - 1 + \frac{1}{2}\int_0^\Delta \beta(\tau)\, d\tau}. \tag{16}$$

In first approximation, if we assume the value of $\beta(\tau)$ to be equal to its limit value of 3/2, we obtain [2] the familiar approximation formula according to which the results of the calculation are in good agreement with the results of the exact solution. As we can see from the exact expression (16), as $\Delta \to \infty$ the approximate and exact formulas coincide ($\beta \to 3/2$).

NOTATION

τ is the optical distance along direction s; $\varkappa$ is the absorption coefficient; $d\omega$ is the solid angle element; ϑ is the acute angle between direction τ and arbitrary direction for hemispheres ($+2\pi$) and (-2π); $I_+(\tau, \vartheta)$ and $I_-(\tau, \vartheta)$ are the radiation intensities in opposite directions; q_+ and q_- are the hemispherical radiant fluxes in opposite directions; q is the net radiant flux; p is the total radiant flux; V is the radiation density; α, β, γ, m_+, and m_- are additional functions; T is the temperature; c is the light velocity; σ is the Stefan-Boltzmann constant; B is the black-body radiation intensity at thermodynamic equilibrium; Δ is the optical thickness; A_b and B_b are the emissivity and radiation intensity of a conventional black surface (at $A_b = 1$); $B^{(k)}(\tau)$ is the k-th order derivative of $B(\tau)$; $E_\nu(x)$ is the integroexponential function of ν-th order; b, k, and ν are subscripts.

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